Impedance Transformation in Folded Dipoles*

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Summary—**It is pointed out that the impedance of a folded dipole relative to that of a simple dipole can be adjusted by employing conductors of different diameters for the separate elements of the folded dipole. Increased impedance ratios can be obtained by the use of additional elements.**

It is shown that the impedance ratio can be obtained from the current ratio and suitable expressions are derived. Practical examples are given.

I. The Folded Dipole as an Impedance Transformer

Any folded dipole[1,2] has a higher impedance at the input terminals than a simple dipole at the same place in any antenna or antenna array. This property of impedance transformation explains the increasing use of folded dipoles, especially at very-high frequencies.

The simplest folded dipole comprises two conductors of equal diameters (see Fig. 1) and gives a step-up

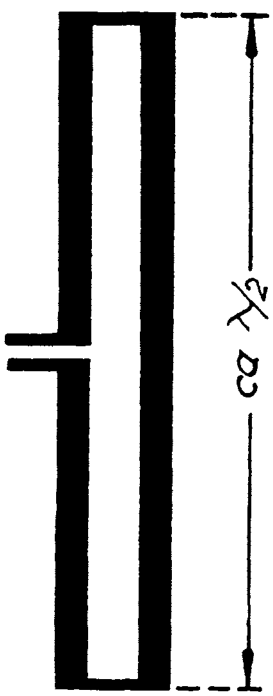

Fig. 1—Folded dipole of two elements of equal diameter.

impedance transformation of 4 to 1. By employing elements of different diameters as in Fig. 2, any desired step-up transformation ratio can be achieved.

If a high transformation ratio is desired, it is practicable to use more than two elements with parallel axes although they need not be in the same plane. A practically important application of three elements is shown in Fig. 3; the axes of the three conductors are in the same plane, the outer elements being identical with each other but generally different from the fed middle element from which they have equal separation.

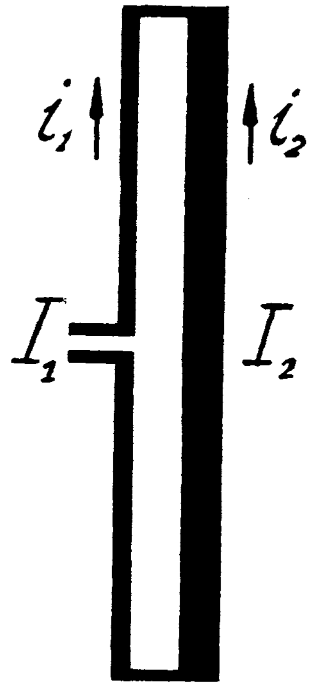

Fig. 2—Folded dipole of two elements of unequal diameter.

Assuming that the radiation from a folded dipole does not differ much from that of a simple dipole at the same place,[3] it is possible to compute the transformation ratio, and consequently the impedance, at the feeding point if the ratio of the currents in the elements of the folded dipole is known.

In Fig. 2, the current (root-mean-square) at the feeding point is designated by I_1 and the current in the center of the auxiliary element by I_2. It is assumed, for simplicity, that the dimensions of the dipole "match" the frequency so that the input impedance is real.

In any array in which the fed element is a simple dipole, let the input resistance at the feeding point be R_0. When the simple dipole is replaced by a folded dipole, let the new input resistance be R_1.

Then, with the above-mentioned assumption of equal radiation, we have the following relation:

$$I_1^2 R_1 = (I_1 + I_2)^2 R_0. \tag{1}$$

The folded dipole therefore gives the resistance transformation ratio u, where u is given by the following expression:

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[1] P. S. Carter, "Simple television antennas," *RCA Rev.*, vol. 4, p. 168; October, 1939.

[2] J. D. Kraus, "Multi-wire dipole antennas," *Electronics*, vol. 13, p. 26; January, 1940.

[3] R. W. P. King, H. R. Mimno, and A. H. Wing, "Transmission Lines, Antennas and Wave Guides," McGraw-Hill Book Co., New York, N. Y., 1945; p. 224.

$$u = R_1/R_0 = [(I_2/I_1) + 1]^2 = (n+1)^2 \tag{2}$$

in which n is the current ratio given by

$$n = I_2/I_1. \tag{3}$$

We can state the resistance transformation if we know the current ratio. The computation of the current ratio is the object of the following sections.

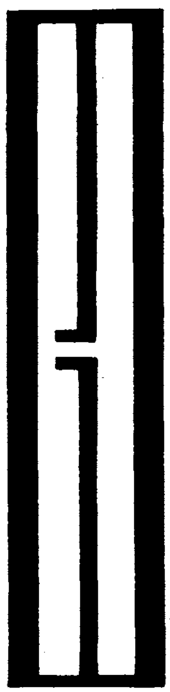

Fig. 3—Folded dipole of three elements.

II. Comparison of a Folded Dipole with a Simple Dipole of Equal Configuration

Consider the dipole of Fig. 4, which is physically like the folded dipole of Fig. 2, except that the auxiliary element is broken and fed in parallel with the first element. The electrical difference is mainly this, that in Fig. 2 out-of-phase "line" currents are superimposed on the in-phase "antenna" currents i_1, i_2. Since the out-of-phase "line" currents are negligible compared to the "antenna" currents at the center points and at the feeding point of the dipole elements,[3,4] we do not need to consider them. Consequently we shall calculate the current partition in a simple unfolded dipole, comprising two or more conductors in parallel as in Fig. 4. The result will be an approximation suitable for engineering design of folded dipoles.

III. The Field Equations in Four-Dimensional Form as a Basis for the Investigation of Folded Dipoles

To describe the electromagnetic relations in the antenna of Fig. 4, we start with the field equations. To enjoy the advantages of more concise expression we shall use them in the four-dimensional representation.[5–8] In addition, we shall choose the potential form which is accentuated by the problem—computation of current and charge distribution.

The four-potential is designated by $\mathbf{\Phi}$, the four-current by $\mathbf{P}$, the six-vector of the electromagnetic field by $\mathbf{F}$. Div and Curl are differential operations which may be represented by the four-dimensional differential operator

$$\diamond = k_1\frac{\partial}{\partial x_1} + k_2\frac{\partial}{\partial x_2} + k_3\frac{\partial}{\partial x_3} + k_4\frac{\partial}{\partial x_4}$$

where $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $x_4 \equiv jct$,
and $\boldsymbol{k}_i$ are the unit vectors of the four-dimensional space.

$$\square = \diamond\cdot\diamond = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}$$

is the four-dimensional form of the Laplace operator

$$\nabla^2 = \Delta.$$

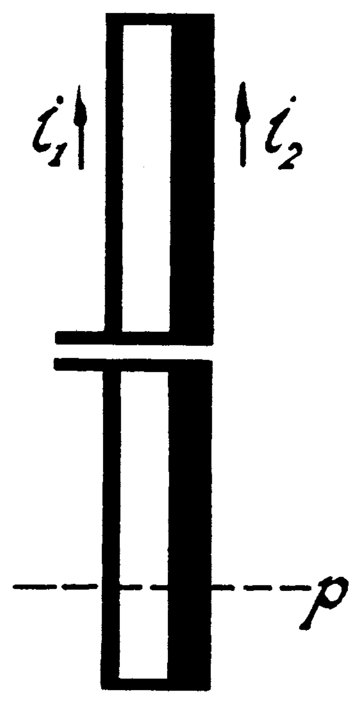

Fig. 4—Simple dipole of same physical elements as the folded dipole of Fig. 2.

[4] W. van B. Robert, "Input impedance of a folded dipole," *RCA Rev.*, vol. 8, p. 289; June, 1947.

[5] A. Sommerfeld, "Zur Relativitaetstheorie. I—Vierdimensionale Vectoralgebra," *Ann. Phys.*, vol. 32, p. 749; June, 1910.

[6] A. Sommerfeld, "Zur Relativitaetstheorie. II—Vierdimensionale Vectoranalysis," *Ann. Phys.*, vol. 33, p. 649; October, 1910

[7] P. Frand and R. v. Mises, "Die Differential und Integralgleichungen der Mechanik und Physik," F. Vieweg, Braunschweig, Germany, vol. 2, 2nd ed., 1935; p. 767.

[8] L. Page and N. I. Adams, "Electrodynamics," D. Van Nostrand Co., Inc., New York, N. Y., 1940; p. 426.

The field equations are now

$$\mathbf{F} = \text{Curl } \mathbf{\Phi} \tag{4}$$

$$\text{Div } \mathbf{\Phi} = 0 \tag{5}$$

$$\square \mathbf{\Phi} = - \mathbf{P} \tag{6}$$

$$\text{Div } \mathbf{P} = 0. \tag{7}$$

Since we are interested in the current distribution i_1, i_2 on the two conductors of the dipole in Fig. 4, we do not require the knowledge of field strengths so that we can neglect (4) in the further considerations. It is obvious that on both conductors all points of the cross sections placed in plane p orthogonal to the axes of the conductors have equal potential at any moment. We may also assume that the currents are flowing only in the surface of conductors, an assumption permissible for practical purposes. The currents, therefore, have only components parallel to the axes. Thus the four current of an element is

$$\mathbf{P} = (1/c)ik_3 + j\rho k_4, \tag{8}$$

if the z axis is parallel to the axes of the conductors. In the formula i designates the current and ρ the charge for unit length of a conductor. Also the four-potential has in this case two components only,

$$\mathbf{\Phi} = A_z k_3 + j\phi k_4, \tag{9}$$

where A_z denotes the vector potential and ϕ the scalar potential.

The currents will be distributed on both elements in such a way that the above-mentioned condition of equal potentials on the conductor cross sections in any plane p, in Fig. 4, will be fulfilled.

Before starting with the calculation of the retarded potentials,[8,9] we shall consider a further simplification.

As a consequence of the equation of continuity (7) which establishes the time-space conservation of electrical charge, it follows that the ratio of charge densities equals the ratio of currents; i.e., from

$$\mathbf{P}_2 : \mathbf{P}_1 = n$$

follows

$$\rho_2 : \rho_1 = i_2 : i_1 = n. \tag{10}$$

Therefore we can confine ourselves to the calculation of the ratio of charges. The same fact is expressed by (5) which states that the four-potential is "divergence-free."[10] Thus the vector potential of the conductor elements in a plane p is in proportion to the scalar potentials. Consequently we may confine ourselves to the calculation of the scalar potential.

We assume further a sinusoidal distribution of currents and charges along the dipole. In addition it will be obvious from the derivation below that the current ratio is not critically dependent on the current and charge distribution along the antenna. Now we substitute the actual charge distribution on each cylindrical conductor by a line charge in parallel with the axis of the cylinder, which produces as far as possible the same potential distribution.

On the basis of these assumptions and simplifications we are able to compute relatively simply with sufficient approximation for practical purposes the potential determining the current distribution. The calculation is given in the Appendix.

IV. Calculation of Current Ratio and Impedance Transformation in the Two-Element Folded Dipole

In the Appendix, the normalized scalar potential (i.e., for $\rho_{\max} = 4\pi\epsilon_0$ in the element 1) is derived for a point in the proximity of a conductor in the plane p if the plane is placed through an end of the dipole.

The formula reads

$$\phi = \log_e \frac{\lambda^{n+1}}{\delta\delta'^n} - \frac{n+1}{2} \text{Cin } 2\pi \tag{11}$$

where δ and δ' are the distances of the reference point and the axes of the conductor 1 and 2.

The "end effect" has not been considered in this formula. Nevertheless, it produces the same rule for designing as formulas derived for planes p which are not placed through an end of the dipole.

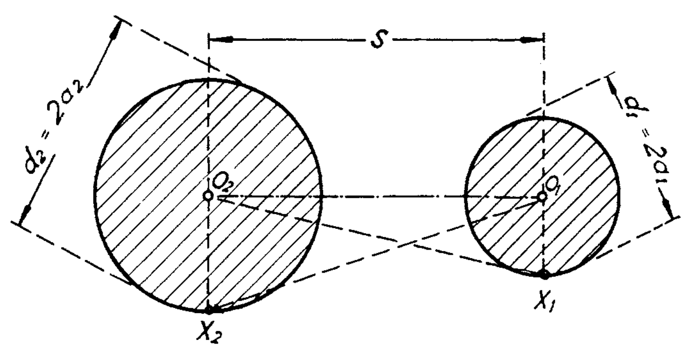

Fig. 5—Cross section through the dipole of Fig. 4 in the plane p.

We employ the formula for computation of the current ratio n if the diameters and the separation of the dipole elements are fixed. As stated above, the potential is the same for all elements of the cross sections of *both* conductors in a plane p. We consider the potential at the points X_1 and X_2 of Fig. 5, and that in a plane p through the end of the dipole at which (11) holds under the idealizations as assumed in Section III. The distance of X_1 from O_2 and of X_2 from O_1 is approximately $O_1O_2 = s$. For the reference point X_1, we must therefore in (11) put $\delta = a_1$, $\delta' \cong s$, and we obtain the potential

$$\phi(X_1) = \log_e \frac{\lambda^{n+1}}{a_1 s^n} - \frac{n+1}{2} \text{Cin } 2\pi. \tag{12}$$

[8] M. Abraham and R. Becker, "Electricity and Magnetism," Blackie and Son, London, England, 1937; p. 220.

[10] H. Minkowski, "Das Relativitaetsprinzip," *Ann. Phys.*, vol. 47, p. 927; September, 1915.

Correspondingly we obtain the potential for X_2 at the same instant if we put $\delta \cong s$ and $\delta' = a_2$

$$\phi(X_2) = \log_e \frac{\lambda^{n+1}}{s a_2^n} - \frac{n+1}{2} \operatorname{Cin} 2\pi. \tag{13}$$

Both values (12) and (13) of the potential must be equal; hence, we obtain immediately

$$a_1 s^n = a_2^n s. \tag{14}$$

The current ratio in question for both conductors is thus approximately

$$n = \log \frac{s}{a_1} \Big/ \log \frac{s}{a_2} \cdot \tag{15}$$

The impedance transformation follows now from (2)

$$u = R_1/R_0 = (n+1)^2 = \left(\log \frac{s^2}{a_1 a_2} \Big/ \log \frac{s}{a_2}\right)^2. \tag{16}$$

These formulas are suitable both for design of folded dipoles if the transformation ratio is fixed, and for computation of the transformation ratio of a given dipole.

Sometimes the following formula obtained from (15) is more convenient

$$n - 1 = \log \frac{a_2}{a_1} \Big/ \log \frac{s}{a_2} \cdot \tag{17}$$

It is immaterial whether we use natural or decade logarithms.

The formulas give a good approximation if

$$a_2/a_1 \geqq 1 \quad \text{and} \quad s/a_2 \geqq 2.5$$

or

$$a_2/a_1 < 1 \quad \text{and} \quad s/a_1 \geqq 2.5.$$

V. Calculation of Current Ratio and Impedance Transformation of a Folded Dipole Comprising More Than Two Elements

As for a two-element dipole we obtain formulas for the design of multi-element folded dipoles by applying (28).

We shall consider only one such type which is of practical interest, namely the symmetrical three-element dipole of Fig. 3, in which the axes of the three elements are in a common plane. If a_1 denotes the radius of the inner element which is fed, a_2 denotes the radius of any of the outer equal elements, s denotes the separation of the inner element from any outer element, and m denotes the current ratio for one outer element to the fed inner element, we obtain approximately

$$m = \log \frac{s}{a_1} \Big/ \log \frac{s}{2a_2} \tag{18}$$

or, more convenient for some problems,

$$m - 1 = \log \frac{2a_2}{a_1} \Big/ \log \frac{s}{2a_2} \cdot \tag{19}$$

The impedance transformation ratio is obviously given by

$$u = (2m+1)^2 = \left(\log \frac{s^3}{2a_1^2 a_2} \Big/ \log \frac{s}{2a_2}\right)^2. \tag{20}$$

Especially it is clear from (18) and (19) that currents in the conductors of a three-element dipole of Fig. 3 are equal, i.e., that $m=1$ only if $a_1 = 2a_2$, that is to say if the inner element has twice the thickness of an outer element. For this case the current ratio is practically (that is, approximately) independent of the separation.

Another specially interesting case is $m=2$, i.e., a transformation ratio $u=25$; according to (19) it is achieved if $s/2a_2 = 2a_2/a_1$, i.e., if the diameter of an outer element is the geometric mean value of the radius of the inner element and the spacing.

The simple approximative formulas (18) to (20) become inaccurate if the separation is too small. The currents and the charges in the outer elements are shifted considerably to the outer parts of the outer elements so that the substituting linear charges ought to be placed some distance outside the axes of the outer elements.

For practical purposes the formulas (18) to (20) may be used for a_2/a_1 between 0.5 and 5, if $s/2a_2 > 2.5$, and for $a_2/a_1 < 0.5$ if $s/a_1 > 2.5$.

VI. Measurements on Two-Element and Three-Element Folded Dipoles

O'Shannassy and Wilkinson performed various measurements on folded dipoles at 150 Mc. They published a part of the results in the literature.[11] Because their measurements are very interesting, the following series are quoted.

Two-Element Folded Dipole

1. Series of measurements (constant spacing):

Dimensions			Transformation Ratio u	
$2a_1$	$2a_2$	s	Measured	Calculated from n of (15)
$\frac{3}{8}''$	$\frac{3}{8}''$	$1\frac{1}{2}''$	3.96	4.0
$\frac{1}{4}''$	$\frac{3}{8}''$	$1\frac{1}{2}''$	4.6	4.8
$\frac{1}{4}''$	$\frac{1}{2}''$	$1\frac{1}{2}''$	6.08	5.7
0.19''	$\frac{1}{2}''$	$1\frac{1}{2}''$	8.3	6.5

2. Series of measurements (constant diameters):

Dimensions			Transformation Ratio u	
$2a_1$	$2a_2$	s	Measured	Calculated from n of (15)
$\frac{1}{4}''$	$\frac{1}{2}''$	$\frac{1}{2}''$	8.89	9.0
		1''	6.19	6.25
		$1\frac{1}{2}''$	5.67	5.7
		2''	5.48	5.44
		$2\frac{1}{2}''$	5.25	5.3

[11] J. O'Shannassy and E. J. Wilkinson, "Some measurements of the impedance multiplication factor of folded dipoles," *Amateur Radio*, vol. 16, p. 7; January, 1948.

Later on O'Shannassy made experiments on three-element folded dipoles which have not yet been concluded since it has proved very difficult to measure high SWR values correctly. The writer therefore quotes from a letter from O'Shannassy only two tentative measurements on symmetrical three-element dipoles (see Fig. 3).

Dimensions			Transformation Ratio u	
$2a_1$	$2a_2$	s	Measured	Calculated from m of (18)
$\frac{1}{4}''$	$\frac{1}{4}''$	$1\frac{1}{2}''$	11.0	14
$\frac{1}{4}''$	$\frac{1}{4}''$	$1''$	12.5	16

VII. Conclusions

The folded dipole of two or more elements is being used as an impedance transformer, e.g., to match an antenna to a line of higher characteristic impedance.

The impedance transformation ratio can be calculated if the current ratio is known. To find this, the two-element folded dipole (see Fig. 2) is compared to a simple dipole of equal physical construction (see Fig. 4), i.e., of two elements in parallel. It is obvious that the charge and current distributions in both types of dipoles are essentially much the same. The charge distribution in the simple two-element dipole can be approximately calculated under the assumption that the (retarded) potentials on coplanar cross sections of both elements are equal.

Measurements seem to prove the practical applicability of the approximate formulas.

VII. Acknowledgments

The author wishes to acknowledge his indebtedness to K. W. Magee (director of Austronic Engineering Laboratories, Melbourne) for the lively discussions which stimulated the investigations[12] which provided the basis for this paper, and to express his appreciation to Standard Telephones & Cables Pty. Ltd. for assistance in preparing the paper for publication.

He also wishes to thank J. O'Shannassy and E. J. Wilkinson of P.M.G.'s Department, Melbourne, for their measurements of which those on three-element dipoles were undertaken at the personal request of the author.

IX. Appendix

Calculation of Retarded Potential in the Proximity of a Folded Dipole

The first step is to draw according to Fig. 6 a system of co-ordinate axes through a folded dipole as in Fig. 2 or a single dipole of the same physical construction as in Fig. 4. The z-axis coincides with the straight line charge equivalent to the fed element 1. The potential at any point X is the sum of the potential of conductor 1 and the potential of conductor 2.

First we shall calculate the potential produced by the charge on conductor 1 only. The charge per unit length of conductor 1 will be designated by ρ, thus ρdz will be the charge of a conductor element of length dz. The distance of the charge ρdz from the reference point X may be called r, the distance of point X from the z-axis may be δ, and ζ denotes the height of X over the xy plane.

The retarded potential[7–9] is given by

$$\Psi = \frac{1}{4\pi\epsilon_0}\int_{z=0}^{\lambda/2} \frac{1}{r}[\rho]dz \tag{21}$$

where $[\rho]$ denotes the retarded charge.

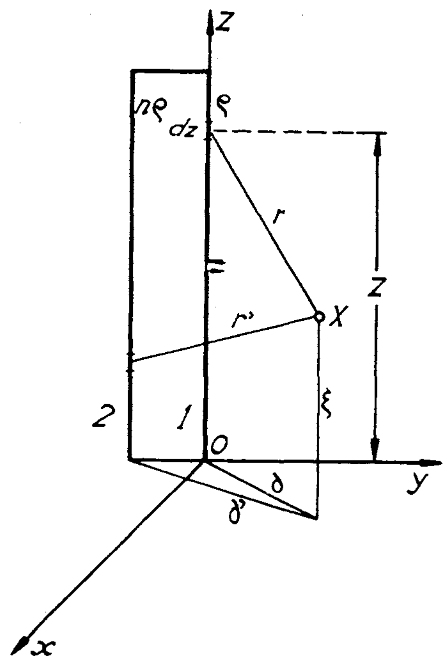

Fig. 6—Folded dipole showing co-ordinate system for calculation of the retarded potential.

If ρ_{max} represents the maximum of charge density ρ with respect to time and space, and if we assume sinusoidal distribution of charge along the dipole, the retarded charge density along conductor 1 is

$$[\rho] = \rho_{max} \cos 2\pi \frac{z}{\lambda} \cos \omega\left(t - \frac{r}{c}\right)$$

or, by suitable choice of zero time, more simply

$$[\bar{\rho}] = 4\pi\epsilon_0 \cos 2\pi \frac{z}{\lambda} \cos 2\pi \frac{r}{\lambda} \tag{22}$$

where

$$[\bar{\rho}] = 4\pi\epsilon_0[\rho]/\rho_{max}$$

[12] K. W. Magee, "Unfolding the folded dipole," *Amateur Radio*, vol. 15, p. 3; May, 1947.

denotes the "normalized" retarded charge which shall be used from now on. By substituting (22) in (21) we obtain the formula for the 'normalized" scalar potential due to conductor 1

$$\psi = \int_{z=0}^{\lambda/2} \frac{1}{r} \cos 2\pi \frac{r}{\lambda} \cos 2\pi \frac{z}{\lambda}\, dz. \tag{23}$$

For convenience all quantities may be considered as measured in electrical angular degrees; i.e., we write now r instead of $2\pi r/\lambda$; z instead of $2\pi z/\lambda$; ζ instead of $2\pi\zeta/\lambda$; and δ instead of $2\pi\delta/\lambda$. The limits have also to be taken in angular scale, so that (23) becomes

$$\psi = \int_{z=0}^{\pi} \frac{1}{r} \cos r \cos z\, dz. \tag{24}$$

Applying the co-ordinate transformation $z-\zeta=u$ we obtain for (24)

$$\psi = \int_{u_1=-\zeta}^{u_2=\pi-\zeta} r^{-1} \cos r \cos (u+\zeta) du$$

$$= \int_{u_1}^{u_2} (u^2+\delta^2)^{-1/2} \cos (u^2+\delta^2)^{1/2} \cos (u+\zeta) du.$$

This we transform by means of the addition theorem of trigonometric functions into

$$\psi = \tfrac{1}{2} \cos \zeta \left[\int_{u_1}^{u_2} \frac{1}{r} \cos (r+u) du + \int_{u_1}^{u_2} \frac{1}{r} \cos (r-u) du \right]$$

$$- \tfrac{1}{2} \sin \zeta \left[\int_{u_1}^{u_2} \frac{1}{r} \sin (r+u) du - \int_{u_1}^{u_2} \frac{1}{r} \sin (r-u) du \right].$$

Introducing new variables for the arguments $(r+u)$ or $(r-u)$, we reduce the individual integrals to the cosine integral or sine integral. Substituting the limits we obtain the potential in numerically calculable form,

$$\psi = \tfrac{1}{2} \cos \zeta [-\mathrm{Ci}\,(r_1+u_1) + \mathrm{Ci}\,(r_1-u_1) + \mathrm{Ci}\,(r_2+u_2) - \mathrm{Ci}\,(r_2-u_2)] + \tfrac{1}{2} \sin \zeta [\mathrm{Si}\,(r_1+u_1) + \mathrm{Si}\,(r_1-u_1) - \mathrm{Si}\,(r_2+u_2) - \mathrm{Si}\,(r_2-u_2)], \tag{25}$$

putting, for simplicity,

$$r_1 = \sqrt{u_1^2+\delta^2}, \qquad r_2 = \sqrt{u_2^2+\delta^2}.$$

Using the well-known definition[13,14]

$$\mathrm{Ci}\,(x) = \gamma + \log_e x - \mathrm{Cin}\,(x)$$

where γ denotes the Euler constant, we can transform (25) in a more suitable form for our purpose

$$\psi = \tfrac{1}{2} \cos \zeta \left[2 \log_e \frac{r_2+u_2}{r_1+u_1} + \mathrm{Cin}\,(r_1+u_1) - \mathrm{Cin}\,(r_1-u_1) - \mathrm{Cin}\,(r_2+u_2) + \mathrm{Cin}\,(r_2-u_2) \right]$$

$$+ \tfrac{1}{2} \sin \zeta [\mathrm{Si}\,(r_1+u_1) + \mathrm{Si}\,(r_1-u_1) - \mathrm{Si}\,(r_2+u_2) - \mathrm{Si}\,(r_2-u_2)]. \tag{26}$$

To simplify as much as possible the following considerations we choose a reference point X in the xy plane; i.e., we put $\zeta=0$. Thus $u_1=0$, $u_2=\pi$, $r_1=\delta$, $r_2=\sqrt{\pi^2+\delta^2}$. Hence, the potential of a point in the xy plane is

$$\psi = \log_e \frac{\pi+\sqrt{\pi^2+\delta^2}}{\delta} - \tfrac{1}{2}\,\mathrm{Cin}\,(\pi+\sqrt{\pi^2+\delta^2}) + \tfrac{1}{2}\,\mathrm{Cin}\,(-\pi+\sqrt{\pi^2+\delta^2}). \tag{27}$$

For a reference point in the proximity of the dipole is $\delta^2 \ll \pi^2$, thus with a good approximation

$$\psi \cong \log_e \frac{2\pi}{\delta} - \frac{1}{2}\,\mathrm{Cin}\,2\pi. \tag{28}$$

Since the charge of the second conductor (see Fig. 6) is n times larger according to (10), the potential produced by the second element at point X is

$$\psi' \cong n\left(\log_e \frac{2\pi}{\delta'} - \frac{1}{2}\,\mathrm{Cin}\,2\pi\right);$$

hence, the total potential

$$\phi = \psi + \psi' = \log_e \frac{(2\pi)^{n+1}}{\delta\delta'^n} - \frac{n+1}{2}\,\mathrm{Cin}\,2\pi.$$

By returning from the angular scale to length scale we have to write again $2\pi\delta/\lambda$ and $2\pi\delta'/\lambda$ instead of δ and δ', so that we obtain finally for the potential in a plane through an end of the dipole of Fig. 4 the formula

$$\phi = \log_e \frac{\lambda^{n+1}}{\delta\delta'^n} - \frac{n+1}{2}\,\mathrm{Cin}\,2\pi, \tag{11}$$

which also holds for the folded dipole of Fig. 2, at least with good enough approximation for practical purposes.

[13] F. E. Terman, "Radio Engineers Handbook," McGraw-Hill Book Co., New York, N. Y., 1943; p. 17.
[14] W. Magnus and F. Oberhettinger, "Formeln und Saetze fuer die speziellen Funktionen der mathematischen Physik," Springer, Berlin, Germany, 1943; p. 97.